Introductory Chapter: Problem Solving Approach to Develop Mathematical Thinking

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In this book, the theory for developing mathematical thinking in the classroom will be explained in Part I and illuminating examples of developing mathematical thinking using number tables will be provided in Part II. For preparation of those two Parts, this chapter briefly explains the teaching approach, called the "Problem Solving Approach," which is necessary to develop mathematical thinking. This chapter describes the approach and explains why it is useful for developing mathematical thinking.

1.1 The Teaching Approach as the Result of Lesson Study

Stigler and Hiebert [1999] explained the Japanese teaching approach as follows:

> Teachers begin by presenting students with a mathematics problem employing principles they have not yet learned. They then work alone or in small groups to devise a solution. After a few minutes, students are called on to present their answers; the whole

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> class works through the problems and solutions, uncovering the related mathematical concepts and reasoning.[1]

In "Before It's Too Late," *Report to the Nation from the National Commission on Mathematics and Science Teaching for the 21st Century*, it was compared with the US approach (2000):

> The basic teaching style in American mathematics classrooms remains essentially what it was two generations ago. In Japan, by contrast, closely supervised, collaborative work among students is the norm.

It was a major document which has led to the current world movement of lesson study, because it informed us about the achievements of the lesson study originated from Japan and recognized it as an ongoing improvement system of teaching by teachers.

The Japanese teaching approach itself was researched by US math-educators through the comparative study of problem solving by Miwa and Becker in the 1980s. In Japan, the Japanese teaching approach which was mentioned above by Stigler and Hiebert is known as the Problem Solving Approach. The comparative study by Miwa and Becker was one of the motivations why the Japanese approach was videotaped in the TIMSS videotape study.

The approach was the result of lesson study in the twentieth century [Isoda *et al.* 2007; Isoda and Nakamura, 2010]. It was known to have been practiced even before World War II but was explicitly recommended after World War II in the national curriculum document of the Ministry of Education. In the 1960s, it was recognized as the teaching approach for developing mathematical thinking which was recommended for developing higher-order thinking for human character formation. For instance, in the late 1960s, the Lesson Study Group of the Attached Junior High School

[1] This Japanese approach was well visualized through the TIMSS videotape study: http://nces.ed.gov/pubs99/1999074.pdf/. The sample videos of the Japanese classroom which are not related to TIMSS can be seen in the following: http://www.criced.tsukuba.ac.jp/math/video/; See Isoda *et al.* [2007].

of Tokyo University of Education (which later changed its name to the "University of Tsukuba") published a series of lesson study books on the approach [Mathematics Education Research Group, 1969].

1.1.1 *Learning mathematics by/for themselves*

The basic principle of the Problem Solving Approach is to nurture children's learning of mathematics by/for themselves. It means that we would like to develop children who think and learn mathematics by/for themselves.

Firstly, we should know how we can learn mathematics by/for ourselves:

Please calculate: $37 \times 3 =$ ____.

When you calculate, do you see any interesting things?

If so, what do you want to do next?

If you can do some activities related to such questions and find something, then you begin to learn mathematics by yourself.

Solve the following:

$37 \times 3 =$ ___

$37 \times 6 =$ ___

___ × ___ = ___

Figure 1.

To nurture children who think and learn mathematics by/for themselves, it is necessary to teach children how to develop mathematics. However, there seem to be only a limited number of people who know how to enjoy mathematics, have a good number sense, and know how to develop mathematics by thinking about the next step. Actually, well-nurtured children, usually given this kind of situation to consider the next step by themselves, can also imagine the next step.

There is no problem even if children cannot imagine the next step for the question "$37 \times 3 = ?$". Then, teachers can devise the task shown in Figure 1 [Gould, Isoda, and Foo, 2010; Hosomizu, 2006, in Japanese] and ask the children to consider the meaning of the blanks and make the questions such as "What do you want to fill in those empty spaces with?" and "Anything unusual or mysterious?". If the children have an idea of what they would like to do next, give them

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the opportunity to do it. If the teacher gives them the time, some children may be able to find something even if other children may not. The children who have found something usually show it in their eyes and look at the teacher to say something. Please listen to your children's idea and just say: "Yes, it is good!" Then, other children may also show interest, and brighten their eyes.

If not, ask the children to fill in the spaces with "37 × 9" just below "37 × 6" and continue to ask until they will easily imagine the next task. If the children can calculate by themselves, normally many of them will have noticed something fascinating based on their expectations and begin to explain to each other what they have found interesting. If they feel the urge to explain why, then they have been experiencing good mathematics teaching because they know that the explanation of patterns with reason is at the heart of mathematics. Having interest and a sense of mystery, and recognizing further the expectation and imagination regarding what to do next gives rise to situations which present opportunities for children to explore mathematics by/for themselves.

If your children do not show any of these feelings at the moment, you do not need to worry, because that is just the result of past teaching. Now is the best time to teach them what they can do next. If the children learn the way of mathematical thinking and appreciate how simple, easy, fast, meaningful, useful, and enjoyable it is to do mathematics, the next time they may want to consider what they would like to do next in similar situations. Even though your children are having difficulties in calculation, if they recognize the mathematical beauty of the number pattern, it presents the opportunity for them to appreciate the beauty of mathematics which lies beyond calculations. They are able to find the beautiful patterns because they know how to calculate. The higher order mathematical thinking are usually documented and prescribed in one's national curriculum. However the approaches to achieve them are not always described. This monograph aims to explain a teaching approach to developing mathematical thinking based on the appreciation of mathematical ideas and thinking. In Part I, Katagiri explains the importance of developing the mathematical attitude that serves as the driving force of mathematical thinking because mathematical thinking is possible only when children would like to think by themselves.

1.1.2 *The difference between tasks and problems (problematic)*

In the Problem Solving Approach, the tasks are given by the teachers but the problematic or problems which originate from the tasks for answering and need to be solved are usually expected to be posed by the children. In this case, the problematic consists of those things which the children would like to do next. It is related with children's expectations on their context of learning. The problem is not necessarily the same as the given task and depends on the children. It is usually related to what the children have learned before, because children are able to think based on what they have already learned.

If your children begin to think about the next problem for themselves, then enjoy it together with them until they tell you what they want to do next (Figure 2). We would like you to continue until the children come up with an expectation. If the children expect that "555" comes next, then you just ask them: "Really?" In the mathematics classroom, the task is usually assigned by teachers but, through the questioning by the teachers, it becomes the children's problems. It is only then that it is regarded as being problematic by the children. We would like you to change your children's belief from just solving a task given by you to posing problems by themselves in order to learn and develop their mathematics.

Solve the following:

$37 \times 3 = 111$
$37 \times 6 = 222$
$37 \times 9 = 333$
$37 \times 12 = 444$
$37 \times 15 =$ ___

Figure 2.

If you ask, "Why do you think it will be 555?" some possible responses will be "Because the same numbers are lined up," "It has a pattern" and "Because of the calculations...." If the teacher asks "Why?" then the children are given the opportunity to develop their ability to explain why (i.e. to give reasons). Your question “really?” against the children’s prediction involves a number of hidden yet wonderful questions. The way the multiplication results come up as identical digits (as if dice were rolled, resulting in every die landing with the same number up) is itself a mystery. Even before the children do the calculation on paper, they can predict that the next answer will be 555 and, sure enough, that is the answer they get. This is a mystery.

To explain this mystery, look at Figure 3. Every time 3 is added to the multiplier (3, 6, 9,….), the answer increases by 111. This is in spite of the fact that the multiplier has gone up by only 3. Here the arrow ↓ represents the structure well. If the children know the ↓ representation for showing a mutual relationship, it means that they already have the experience to represent the functional relationship such as proportionality or linearity by arrow even if they have not yet learned the term “proportion.” We should use the arrows from the first grade to represent relationships like this. If the children do not know the arrow representations, then the teacher represents what the children have found (3, 6, 9,....) by arrows on the board using yellow chalk with “+3.” If the children have also found “+111” on the arrow ↓ between lines, then ask them to explain other arrows for confirming the proportionality or the same pattern by how they multiplied with repeated additions. Through knowing the relationship between two types of arrows, children may understand proportionality even if they do not know what to call it.

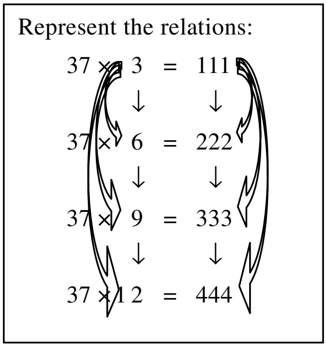

Figure 3.

Once this way of explanation becomes possible, the problem’s significance deepens into “Whenever the multiplier is increased by 3, will the answer always increase by 111, with all the digits identical, the same?” and then “Why are all the digits identical?”.

Readers who are majors in mathematics may already have predicted that this idea holds only “up to 27.” It is true that we get the answer 999, when doing the multiplication 37 × 27.

1.1.3 *Teachers' questioning, and changing and adding representations*

The activity shows the importance of teachers' questioning and representations to promote children's mathematical thinking, which will be re-explained in Part I.

When teachers represent the relationship by arrows, it is possible that children can explain the pattern regarding why a $37 \times (3 \times \_\_)$ involves using the 3s row of the multiplication table for the multiplier. The reason the digits come up to be the same is that this is $37 \times (3 \times \_\_) = 37 \times 3 \times \_\_$, and $37 \times 3 = 111$, and so this can be explained as being the same as $111 \times \_\_$. This is the chance to recognize that we can explain patterns based on the first step of the pattern.[2]

It is interesting to see how what one has already learned in mathematics can be used to explain the next ideas. Using what we here learned/done before is one of the most important reasonings in mathematics. To recognize and understand the reason, the arrow representation is the key in this case. Since the arrow representation makes it possible to compare the relationship between mathematical sentences. To understand and develop mathematical reasoning, we usually change the representation for an explanation in order to represent mathematical ideas meaningfully and visually. It is also a good opportunity for children to experience a sense of relief upon finding the solution to this mystery using the idea of the associative law. Even if they do not know the law, they will understand well the significance of changing order in multiplication.

1.1.4 *Extending the ideas which we have already learned*

Actually, the identical digits pattern comes to an end after the multiplication by 27. Do we then learn anything else by continuing the calculations beyond 27?

[2] For recognition like this, we should develop children who can read an expression in various ways. This will be explained in Part I.

When one starts from "for example," one begins to recognize a new pattern.

$37 \times 30 = 1110$
$37 \times 33 = 1221$
$37 \times 36 = 1332$
$37 \times 39 = 1443$
$37 \times 42 =$ ____

Figure 4.

The pattern is that "the tens digits and the hundreds digits are identical." Not only that, but when one looks a little closer, one can see that "the tens and hundreds digits are equal to the sum of the ones and thousands digits." In other words, in the case of "1, 3, 3, 2," 1 + 2 = 3. "That's crazy — how can this be?" There is a sense of wonder inspired by this, and the questions "Is this really true?" and "Does this always hold true?" lead us to ask: "Why?"

Beyond this point, one must do some calculation. By actually doing the multiplication in vertical form on paper instead of adding 111, one starts to see why this pattern works the way it does. Changing and adding representations are usually the key to new ways of explanation.

Since $37 \times 36 = 37 \times (3 \times 12) = (37 \times 3) \times 12 = 111 \times 12 = 111 \times (10 + 2) = 1110 + 222$, the tens digit and the hundreds digit must be identical, and this digit will be the sum of the thousands digit and the ones digit. This identical digit is derived through $37 \times (3 \times \underline{\quad})$, as the sum of the tens digit and the ones digit in the blank.

Then, we have established the new pattern, haven't we? It is interesting that this idea can be seen as an extension of the previous idea. Indeed, 999 is 0999. "0, 9, 9, 9" is "0 + 9 = 9." Then, $37 \times 27 = 37 \times (3 \times 09) = (37 \times 3) \times 09 = 111 \times 09 = 0000 + 999 = 0999$. So the two different patterns can be seen as single pattern.[3] But how far does this pattern hold? There is no end to the activities one can carry out while pursuing the enjoyment of mathematics in this way.

Mathematicians such as Devlin [1994] have characterized mathematics as the patterns of science. From the viewpoint of the mathematical activities, the activity of completing a given task is no more than what is given. As one completes the task, one discovers a fascinating phenomenon — namely, the existence of invariant patterns

[3] In Part I, Katagiri calls it "integrative thinking."

amidst various changes. While examining whether or not that pattern holds under all circumstances, or when it holds, one discovers mathematics that was previously unknown. By applying what one has learned previously in order to take on the challenge of this kind of problem, not only can one solve the problem, but it is also possible to experience the real thrill and enjoyment of mathematics.

If you do not believe that teachers can develop children's mathematical thinking, solve the following task with the children:

$$15873 \times 7 =$$

This task appeared in the *Journal of Mathematics Education for Elementary Schools* (1937, p. 141; in Japanese). This was one of the journals of lesson study in mathematics before World War II. We can imagine a number of children who will be challenged to move to the next step by themselves because if they can calculate $15873 \times 7 = 111111$, they may begin to think that it is a similar problem. From the similarity, they can think of next step.[4] If the children who have learned from 37×3 can pose a new challenge by/for themselves, it means that they have learned from the previous activities on 37×3.[5] If the children can create expectations of the next step by themselves, it means that they have learned how to learn from the learning process. This is the way to develop mathematical thinking.

1.2 Setting the Activities for Explaining, Listening, Reflecting, and Appreciating in Class

To teach mathematics with these kinds of mathematical activities, we do not only ask children to solve the task given by the teacher, but also give the children opportunities to consider what they would like to do next based on their expectations. Ask them to solve their problems, and listen to their exploration and appreciate their activities as they

[4] In Part II, it will be called "analogical thinking."

[5] If not, the previous activities are not done by the children but by the teacher as a kind of lecture aimed at teaching the result, not the process, or the teacher has failed to give the opportunity where the children can learn how to learn mathematics from the reflection on experience based on appreciation.

begin to learn how to develop mathematics for themselves. If the children reflect on their activities in specific situations, recognize the thinking that was necessary for developing the mathematics from that experience, and appreciate it well, the children may have a wish to use it in other, similar situations. This is the way to nurture and to develop children who can do what they have learned from the experience.

The problem-solving approach, which was mentioned by Stigler and Hiebert [1999], is the teaching approach used to enhance the learning from these processes.

1.2.1 *Structure of Problem Solving Approaches*

The Problem Solving Approach is the method of teaching used to teach content such as mathematical concepts and skills, and mathematical process skills such as thinking, ideas, and values. It follows the teaching phases as in Figure 5.

The phases are a model and need not be followed exactly because a teacher manages the class for the children depending on his/her objective, the content, and the understanding of the children. It is also not necessary to apply all these phases in one

Phase	Teacher's influence	Children's status
Posing the problem	*Posing the task with a hidden objective*	Given the task in the context but not necessary to know the objective of the class.
Planning and predicting the solution	*Guiding children's to recognize the objective*	Having expectations, recognizing both the known and the unknown, what are really problems (including the objective of the class) and their approaches.
Executing solutions/ independent solving	*Supporting individual work*	Trying to solve for having ideas. For preparing explanations, clarifying and bridging the known and unknown in each approach, and trying to represent better ways. If every child has ideas, it is enough. (Do not wait until all the children give correct answers, because answering is not the main work for the class. While waiting, children lose ideas and hot feeling, which should be discussed.)
Explanation and discussion/ validation and comparison	*Guiding discussion based on the objective*	Explaining each approach and comparing approaches based on the objective through the bridging between the known and the unknown by all. (This communication for understanding other ideas, considering other ways, and valuing is the main work for the class.)
Summarization/ application and further development	*Guiding the reflection*	Knowing and reorganizing what they learned through the class and appreciating their achievement, ways of thinking, ideas and values. Valuing again through applying ideas.

Figure 5. Phases of the class for Problem Solving Approach.

teaching period. Sometimes the phases can be applied over two or three teaching periods. Furthermore, the teacher does not need to follow these phases in cases where exercises are given to develop the children's calculation skills. Even though there are variations, the phases are fixed for explaining the ways to develop mathematical thinking in class. Otherwise, it is difficult to explain the teaching approach even if we choose it depending on the necessity.

The phases for teaching do not mean that teachers have to teach mathematics step by step. For example, the phase of independent solving does not mean that all children have to solve the task in this phase even if the teacher supports the children's work. Children who cannot solve the given task can learn from their friends how to solve it in the phase of explanation and discussion. At the end, children who have failed still have the chance to apply learned ideas to the task for further development. Basically, before the class, the teachers develop the lesson plans for supporting the children in each phase and set the decision-making conditions for observing, assessing, and supporting the children. In Part II, the assessment to develop mathematical thinking in the teaching process is explained for each lesson plan.

1.2.2 *Diversity of solutions and the objective of the class*

Now, solve the next three tasks in Figure 6. There are a number of solutions, depending on what the children have already learned.

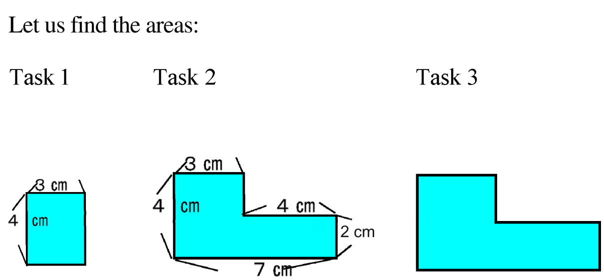

Figure 6.

In case the children have learned the formula of the area of a rectangle and have not yet learned the area of other figures such as a parallelogram, Task 1 is just an application of the formula. Task 2 is a problem in case they do not know to see the shape as the composition or decomposition of two rectangles. In Task 3, the children must start by measuring lengths. The area will change depending on the values of the measured sides. Then:

Task 1 has one solution method and one correct answer;

Task 2 has various solution methods and one correct answer;

Task 3 has various solution methods and may have more than one correct answer.

If one includes incorrect answers, each of these problems will have more than one answer. In particular, tasks with multiple solution methods and multiple correct answers such as the one shown in Figure 3 are sometimes referred to as open-ended tasks [Becker and Shimada 1997/1977]. Depending on what the children have already learned, the diversity itself changes. If the children have only learned that the unit square is 1 cm^2, then Task 1 has various solutions.

In the Problem Solving Approach, tasks and problems are usually set depending on the curriculum sequence. The curriculum, such as the textbook, describes today's class between what the children have already learned and how they will use the idea in a future class which should be taught in today's class. The objective of teaching is usually defined in the curriculum sequence.[6] In the Problem Solving Approach, a task which has various possible solutions is posed for children to distinguish between what is learned and what is unknown: Here the term "unknown" refers to the aspects that have not yet

[6] Normally, the objective includes both content and process objectives. The objective is recommended to be written in the following format: "Through the process, learn the content" or "Through the content, learn the process." The teaching of process skills such as ways of thinking, ideas, and values is warranted by this format. In the lesson study, the teacher is expected to explain why he or she chose the subject matter based on both content and process objective. In the following discussion with the case of Task 2, the objective is: Through exploring how to calculate the area of Task 2, children learn about the permanence of area such as by addition and subtraction.

been learned rather than the answer for the particular task, itself. To solve the task, the children have to make the unknown understandable. This is the planned problematic for the children set by the teacher. What this means is that this planned problematic is hidden in the specified task and corresponds to the teacher's specified teaching objective for the specified class in the curriculum sequence.

For example, if there is a curriculum sequence in which the children learn the additive property of area (the invariance of the area when the shape changes) after the formula for the area of a rectangle has been learned, Task 2 is better than Task 3. This is because Task 2 allows various answers such as addition and subtraction of different rectangles to be compared. From the comparison, the children learn about the permanence of the area in the addition and subtraction of areas. Using Task 3 it is impossible to compare the different answers for this objective, because the difference originates from the ways and results of the measurements. Thus, the objective of the task is fixed according to the curriculum sequence and the conditions of the task are controlled by teachers who will teach today's class based on their objectives.

1.2.3 *Comparison based on the problematic*

The children's problematic is the objective of teaching from the viewpoint of the teachers.

After solving the task, the teacher calls the children to present their ideas in front. The children begin to explain. The teacher just praises the children if the children find their solutions and then begin to lecture on what they want to teach. These classes are usually seen at the challenging stage of an open-ended approach. They are very good and better than just a lecture, because the children are given the opportunity to present their ideas. On the other hand, if the teacher just explains his order understanding without relating it to the children's presented ideas, the children cannot connect what they already know and the teacher's explanation. Nor can they summarize what they have learned today. Presentations of various solutions are necessary but the key is the comparison of the differences from the viewpoint of the problematic in order to achieve the objectives.

In the case of Task 2, if the children recognize the problematic in finding the area of a figure which is not a rectangle, we can compare solutions such as by just counting the number of unit squares, adding two rectangles and subtracting the unseen rectangle from the large rectangle: the figure is a combination of the unit squares, the figure is a combination of rectangles, the figure is part of one large rectangle. Through the comparison the children recognize these differences. Depending on how the children recognize the figure as a component of squares and rectangles, their answers will be different but the result will be the same. From the children's explanation, the teacher draws a conclusion on the invariance of the area through the addition and subtraction of figures. Through comparison, the teachers enable the children to reflect on their activities. This conclusion is possible only through a diversity of solutions from the children and is not achieved through an individual solution from each child. What this means is that the Problem Solving Approach is aimed not only at getting the answer for the given task but also at developing and appreciating the mathematical concept, general ideas of mathematics, and the ways of thinking through exploring the problematic posed by the children, which is related to the objective of teaching.

1.2.4 *Using the blackboard for illustrating children's thinking process*

Another key to the Problem Solving Approach is the ways of using the blackboard (whiteboard) to allow children to learn mathematics for themselves. Japanese elementary school teachers use the board based on the ideas of the children and the children's presentations, and do not erase the board to allow the children to reflect during the summing-up phases toward the end of the class. Figure 8 shows a sample format of the blackboard [Isoda *et al.*, 2009], and Figure 7 presents a case of the area of a trapezoid. The blackboard is not intended to be used to write what the teacher wants to teach but to show how the class is going to learn from the children's ideas.

Figure 7. The case of the area of a trapezoid: the left photo shows the independent solving phase; the middle and right photos shows the phase of explanation. The children are explaining how to calculate the area of a trapezoid using what they have learned before. At the previous grade, they have already learned the area formula of a rectangle and that the area is not changed by addition and subtraction. In the middle photo, there are four presentation sheets, show the back sides, which are not yet presented. For comparison, the teacher chose the presenters based on her consideration of the order of the presentations during the independent solving phase.

It shows the process of all the class activities. It enables the children to reflect on what had happened during their learning process, whose ideas were presented, which ideas were similar, how the ideas were evaluated by the child and his or her friends, and what the lesson can achieve from these learning processes.

1.3 The Roles of the Curriculum and Textbooks

The Problem Solving Approach is preferred for teaching content and process in order to learn how to learn. This means that the approach has been used to develop mathematical thinking. Parts I and II of this book were originally written by Katagiri in Japanese and edited and translated into English by Isoda. In Part I, Katagiri's proposed approach does not explain the approach, because the approach itself has already been shared in Japan. Illuminating examples in Part II will support one's understanding of this approach. In order to show how to develop mathematical thinking in Part II, we have chosen only the case of the number table as an example for the problem situations from a number of his examples. This is because it is easier to explain the preparation that

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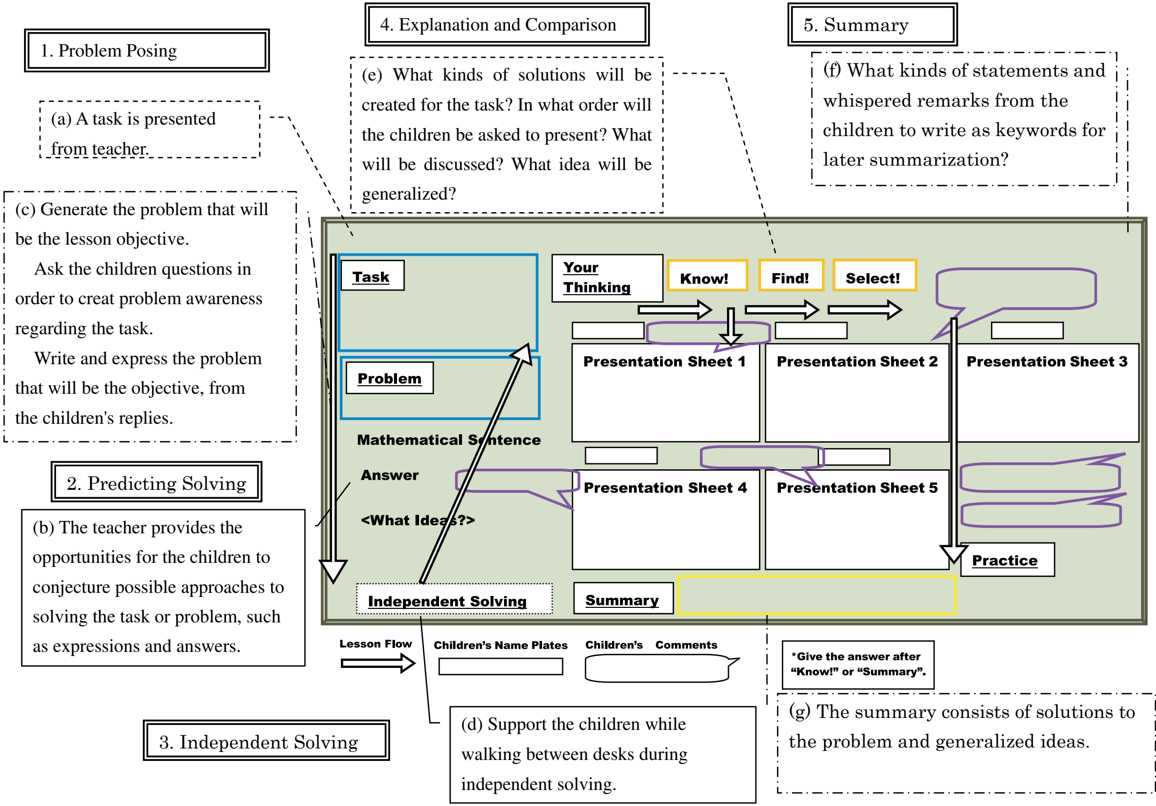
1. Problem Posing
(a) A task is presented from teacher.
(c) Generate the problem that will be the lesson objective.
Ask the children questions in order to creat problem awareness regarding the task.
Write and express the problem that will be the objective, from the children's replies.
2. Predicting Solving
(b) The teacher provides the opportunities for the children to conjecture possible approaches to solving the task or problem, such as expressions and answers.
3. Independent Solving
4. Explanation and Comparison
(e) What kinds of solutions will be created for the task? In what order will the children be asked to present? What will be discussed? What idea will be generalized?
5. Summary
(f) What kinds of statements and whispered remarks from the children to write as keywords for later summarization?
Task
Problem
Mathematical Sentence
Answer
<What Ideas?>
Independent Solving
Your Thinking
Know!
Find!
Select!
Presentation Sheet 1
Presentation Sheet 2
Presentation Sheet 3
Presentation Sheet 4
Presentation Sheet 5
Practice
Summary
Lesson Flow
Children's Name Plates
Children's Comments
*Give the answer after "Know!" or "Summary".
(d) Support the children while walking between desks during independent solving.
(g) The summary consists of solutions to the problem and generalized ideas.

Figure 8.

is necessary for children to engage in so that they can think by themselves during the learning sequences.[7]

Indeed, in the previous task, $37 \times 3 = 111$, children can learn how to develop mathematics and then solve the task $15873 \times 7 = 111111$. In this sequence of teaching, children can explore the second task by themselves if they have learned how to in the first task.

This means that the Problem Solving Approach can possibly be used when the children are well prepared through learning the specific curriculum sequence. In mathematics education research, children's problem solving is sometimes analyzed for the cognitive process to describe how they arrive at the solutions. It is important to know what heuristics is. On the other hand, the Problem Solving Approach is the method of teaching for achieving the preferred objective of teaching. The objective is usually preferred in the curriculum sequence. The subject matter is fixed by the objective. Children can learn future content based on what they have learned before. Teaching today's content usually also means preparing children's future learning of mathematics and not just teaching the content for that day. The basic principle of learning mathematics is that children should learn by/for themselves; in every class we teach the methods of developing mathematics, mathematical ideas, and its values for children's further learning. By teaching mathematical thinking consistently, we can prepare children to think by/for themselves.

To teach mathematical thinking consistently, the Japanese have developed elementary school mathematics textbooks. Katagiri's original books written in Japanese include a number of examples that show the dependence on the sequences and selected representations in the Japanese textbooks and curriculum. Readers may not know them, and thus, in this book, the editor has only selected the example of number tables in Part II.

The Japanese textbooks series for elementary schools was developed based on the Problem Solving Approach; see, for example

[7] If we prefer the task which depends on the curriculum, we have to explain what the children have learned before. Even if we explain it, the curriculum is different, depending on the country.

Gakko Tosho's textbooks [2005, 2011]. In Figure 9 (p. 11, Grade 4, 2005 edition), Task 5 is about the area of the L shape (gnomon). The child may have a question: "I can use the formula if…." This is the problematic, the objective of this class. Next (p. 12), various solutions are shown. All answers are appropriate for Task 5. Then, the teacher can summarize by saying that the area does not change by moving, adding, and subtracting. For the next step in application, the children face the challenge of solving Task 6. Then, they recognize that there are applicable ideas and non-applicable ideas. Takeshi's idea does not work. The children reappreciate what they have learned at Task 5 and learn the applicability of ideas. Testing the applicability to other cases is the viability of mathematical ideas, which was enhanced by von Glasersfeld (1995). The Japanese textbook employs the sequence of extension based on what the children have learned before and teaches the children how to extend mathematical ideas using the sequence for extension.

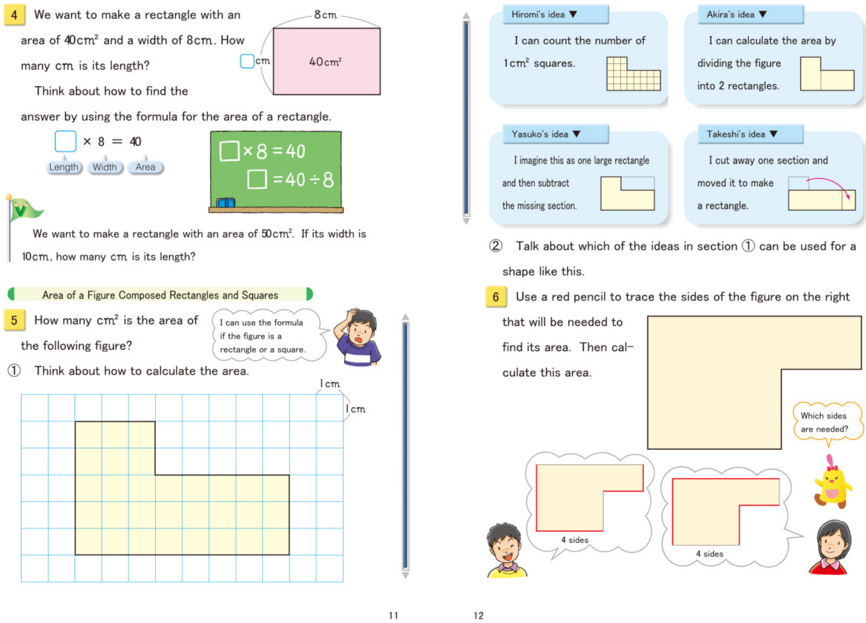

Figure 9. Gakko Tosho's textbook (Grade 4, 2005, p. 11).

Like the sample in Figure 9, generally, Gakko Tosho's textbooks [2006, 2011] have the following features:

- **The preferred Problem Solving Approach for developing children's mathematical thinking:**

  The Problem Solving Approach is preferred for developing children who learn mathematics by/for themselves. The task for the Problem Solving Approach is indicated by a slider mark in the textbook. Normally, the tasks are sited on the odd number pages. At the slider mark, the problematic is written by means of questions from children or the key mark. The children's various ideas are explained on the even number pages, because the children cannot see the even number pages when reading about the task on the odd number page. Through the explanation and comparison of the various ideas, the children are able to learn and the teachers can continue from the various ideas. If you open the textbook, you may recognize a lot of slider marks which have this style. Figure 9 is a good example. All those marks are the result of lesson studies.

- **Using preferred representations in a limited number of pages, and formally and consistently using them to enable children to extend their mathematical ideas:**

  In mathematics education, when we cannot explain what children have learned before, the term "informal" is a good word to explain the children's representations. On the other hand, textbooks select representations and use them formally and consistently as a part of teaching content to support children's mathematical thinking. These formal representations are required for the children to learn further mathematics even if they do not necessarily know it at that moment.

  For example, from first grade to third grade, the block diagram is consistently used for explaining place value. The block is not limited to explaining the base ten system and counting by ones but is also used for counting by multiple base for multiplication.

From second grade to sixth grade, the tape diagram in multiplication is combined with the number line to represent proportionality. This is called the proportional number line. These limited representations are formally and consistently used to enable children to extend four operations and ideas by themselves.

- **Ensuring children's understanding by introducing new ideas through the chapter named "Think About How to Calculate":**

  For developing children's problem-solving skills in multiple ways, some chapters have the previous pages named "Think About How to Calculate," aiming to teach children to think about how to calculate, not the specific way of calculation itself. Through this preparation, children are able to relearn how to use what they have learned before and apply their ideas to unknown situation with necessary representations. This relearning is the preparation for the next chapter. Without this preparation, many children forget what they have learned before, which is necessary for the next chapter.

- **Enhancing the development of mathematics using the method "think about how to…." for enabling children to find their ways, and giving the opportunity to select the methods which can be applied generally:**

  In the Problem Solving Approach, the teaching objective is not just to answer but to develop new ideas of mathematics based on what has been already learned. For the task for the Problem Solving Approach in the textbooks with slider marks, there are questions regarding "think about how to…." which are aimed at showing the recommended problematic for children. By answering these questions, it is hoped that the children do not just get the answer but are also able to find general ideas in mathematics. Based on this problematic question, we can teach children the value of mathematics, which is not limited to solving given tasks but enables children to develop mathematics by themselves.

- **Through dividing one topic into several units and sections, and using recursive teaching–learning ways to teach children learning how to learn:**

There are various dimensions to the learning manner in the mathematics class. For example, any textbook will ask the children to write the expressions for a task. However, at the introductory stage in this textbook, the children do not know the expressions for a particular situation. This textbook carefully distinguishes different situations for each operation. After the ways of calculation are taught, there are sequences to extend the numbers for the calculation. At the same time, we usually ask the children to develop the problem and the story for each operation. And, finally, we introduce the world of each operation for the children to explore the pattern of answers and calculations.

Any textbook will have the sequence to explain the meaning and for the children to acquire skills. Additionally, this textbook adds "think about how to...." questions to enable the children to develop new ways with their meanings and skills. However, this textbook does not have a lot of exercises in the limited number of pages. If necessary, the teacher may be required to prepare some exercises.

If you compare several chapters and sections, you may recognize further ways of learning how to learn. For example, for second grade multiplication, the sections for developing the multiplication table are divided into two chapters. From the multiplication of 2, 5, 3, and 4, the children learn how to develop the multiplication table and then they apply the ways of learning to the next chapter for the multiplication of 6, 7, 8, and 9. In third grade, for Chapter 1, on addition and subtraction, there are questions for planning how to extend the vertical calculation algorithm into large numbers.

If you carefully read the end of chapters, you may find some parts which just aim to teach children learning how to learn and value mathematics. For example, in third grade, (p. 31), there are explanations of how to use the notebook with the questions such as "What do you want to do next?". It means that this textbook attempts to develop children's desire to learn by themselves. On the format of the notebook, which is explained in the textbook, children can learn how to write the explanation with various representations and how to evaluate other children's ideas.

Japanese textbooks such as Gakko Tosho have these features. In particular, only Japanese textbooks contain well-explained children's ideas, even if some of them are inappropriate because they will appear in the classroom. This is the evidence that they are the products of lesson study.

1.4 Perspectives for Developing Mathematical Thinking

To know Katagiri's theory, it is better to be familiar with the several perspectives for developing mathematical thinking which are well known in mathematics education researches. Many of the researches have been done based on their own research questions through case studies by observing limited children in the context of social science. Those researches are out of the scope of this book, because this book is aimed at explaining how to design classroom practice to develop mathematical thinking. To give a clear position to Katagiri's theory, which has been used in the context of classroom practice and lesson study for developing mathematical thinking, here we would like to present some bird's-eye views of the theory.

1.4.1 *Mathematical thinking: a major research topic of lesson study*

In the National Course of Study in Japan, mathematical thinking has been continually enhanced since the 1956 edition. There have been several influences the development of the curriculum before and after World War II, such as the contribution of S. Shimada, who developed the textbook for mathematization in 1943, and the contribution of Y. Wada, [Ikeda, 2010; Matsuzaki, 2010; Mizoguchi, 2010]. Since the 1956 edition of the curriculum, mathematical thinking has been a major aim of mathematics education in the national curriculum.

Katagiri's theory of mathematical thinking began in the 1960s and was mostly completed by the 1980s, and his lesson study groups have been using his ideas since 1960s, until today. If you are involved in research, you may feel that it is an old theory for your reference as it is necessary to refer to the newest articles for research, but in the context of lesson study it serves as the theory that has been approved and used by a great number of teachers in the last half-century. Teachers consequently prefer this theory because of the many evidences that they experience in the process of developing children's mathematical thinking in their classrooms. Many of these experiences are well explained by this theory. He has published 81 books in Japanese for teachers to explain how to develop mathematical thinking. He is still writing. His theory was translated into Korean and he has been working with Korean teachers, too.

Until the 1970s, many math educators in Japan analyzed mathematical thinking for denotative ways of teaching it with specified content in the curriculum even if the national curriculum preferred the connotative ways of explanation. A number of types of mathematical thinking were explained by many researchers. One of his contributions led to this movement and he composed it based on the importance of teaching and making it understandable to teachers even if they are not math majors.[8] Another of his contributions was his ways of composition. In Part I, he composes them based on "mathematical attitude," "ways of thinking," and "ideas." He explains that "mathematical attitude is the driving force of mathematical thinking because we aim to develop children who would like to think by themselves. This means that the child has his or her own wish to explore mathematics. Thus, developing the attitude of thinking

[8] By doing so, mathematical thinking can possibly be learned by elementary school teachers who teach children mathematical thinking. When the teachers plan the class, they can read the curriculum sequence from the viewpoint of developing mathematical thinking consequently. Even though Japanese textbooks have the specific sequence to teach learning how to learn, developing representations and thinking mathematically, if the teachers cannot recognize it, they usually just try to teach skills which they can teach without preparation. If the teachers think mathematically when reading the textbook, they can prepare the year plans to develop the children's mathematical thinking with the use of the textbooks.

mathematically is essential. Mathematical ideas can be typed more deeply. However, he selected major mathematical ideas for elementary school mathematics. This is deeply related to the Japanese tradition of teaching mathematics which enhances the appreciation of mathematics [Isoda, Nakamura, 2010; Makinae, 2011]. Explaining mathematical thinking with the attitude is another contribution of Katagiri.

1.4.2 *Mathematical thinking: a bird's-eye view*

In mathematics education research, there are two traditional references for describing mathematical thinking: one is focused on the mathematical process and the other on conceptual development.

The well-known references of the first type are the articles of Polya [1945, 1957, 1962, 1965]. He analyzed his own experience as a mathematician. His book was written for people challenged by the task given by him. To adopt his ideas in the classroom, teachers have to change the examples to make them understandable and challenging for their children. Mason [1982] refocused on the process from the educational viewpoints. Stacey [2007] described the importance of mathematical thinking and selected twin pairs of activities — "specializing and generalizing" and "conjecturing and convincing" — as follows:

> Mathematical thinking is an important goal of schooling. Mathematical thinking is important as a way of learning mathematics. Mathematical thinking is important for teaching mathematics. Mathematical thinking is a highly complex activity, and a great deal has been written and studied about it. Within this paper, I will give several examples of mathematical thinking, and to demonstrate two pairs of processes through which mathematical thinking very often proceeds: Specialising and Generalising; Conjecturing and Convincing. Being able to use mathematical thinking in solving problems is one of the most the fundamental goals of teaching mathematics, but it is also one of its most elusive goals. It is an ultimate goal of teaching that students will be able to conduct mathematical investigations by themselves, and that they will be able to identify where the mathematics they have learned is applicable in

real world situations. In the phrase of the mathematician Paul Halmos (1980), problem solving is “the heart of mathematics”. However, whilst teachers around the world have considerable successes with achieving this goal, especially with more able students, there is always a great need for improvement, so that more students get a deeper appreciation of what it means to think mathematically and to use mathematics to help in their daily and working lives.

The second focus is on the conceptual development of mathematics. Freudenthal [1973] used the word “mathematization” for considering the process to objectify mathematical activity. What is interesting for researchers is that he said that Polya did not explain mathematical activity. Tall described the conceptual development with the word “procept,” and also described the three mental worlds of embodiment, symbolism, and formalism [Tall and Isoda, to appear]. His map of mathematical thinking, in Figure 10, shows us one bird’s-eye view.

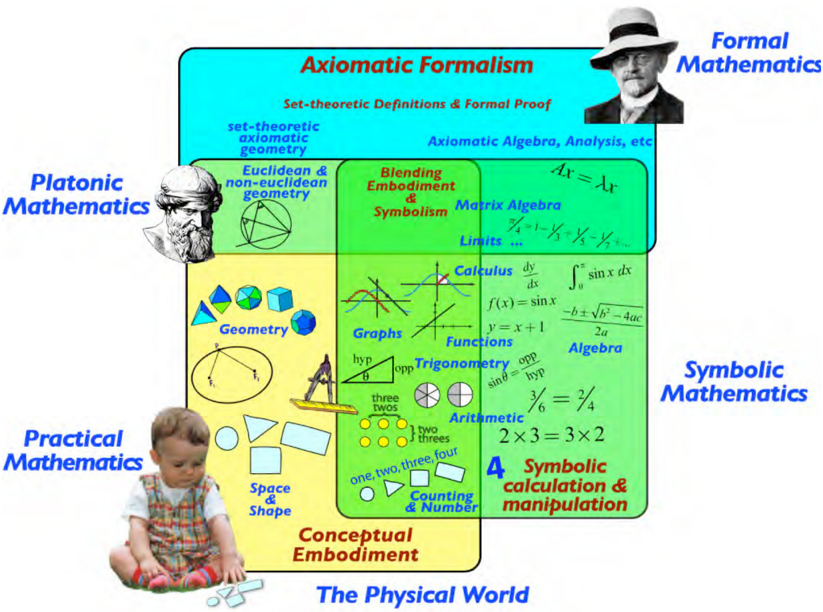

Figure 10. Three Worlds of Mathematics.

These two perspectives of mathematical thinking explain the complex thinking in each thinking process in mathematics in simple terms. Making clear those terms is necessary in orders to know what mathematical thinking is. Each of them shows a kind of denotative description of mathematical thinking.

Additionally, in the last twenty years, there have been curricular reform movements that were focused on competency. New terms are used that are related to mathematical thinking. "Disposition" is one of the words that are well known [Kilpatrick *et al.*, 2001]. It is deeply related to knowing the value of mathematics and the mindset for mathematics.

These major trends in the mathematical process, conceptual development, and dispositions are deeply related to Katagiri's thoughts about mathematical ways of thinking, ideas, and attitude, which will be explained in Part I. As in mathematics education research, it is necessary to clarify the relationships between those key terms, which were explained by Katagiri himself in his previous books written in Japanese in the 1980s. Part I presents just the essence of his theory. At the same time, his view of mathematical thinking will still be considered innovative in mathematics education research, because it is well related to the current ideas about mathematical thinking which have been used in the major research, articles on mathematics education as an academic discipline and, now, lesson study is developing a new research context which recognizes the theory of mathematics education as with reproductive science in classrooms in various settings.

In this introductory chapter, the Problem Solving Approach is only explained briefly, in order to understand Katagiri's work in Parts I and II. The details of the approach will be further explained with a number of evidences of lesson study in further monographs in this series.

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